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OF

TWO-DIMENSIONAL ORTHOGONAL

CURVILINEAR COORDINATES

IN AN

EUCLIDEAN SPACE

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ENGINEERING & INDUSTRIAL RESEARCH STATION
Department of Aerospace Engineering

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In this paper a non-iterative method for the numerical generation of orthogonal curvilinear coordinates for plane annular regions between two arbitrary smooth closed curves has been developed. The basic generating equation is the Gaussian equation for an Euclidean space which has been solved analytically. The method has been applied in many cases and these test results demonstrate that the proposed method can be readily applied to a wide variety of problems.

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NUMERICAL GENERATION OF TWO-DIMENSIONAL ORTHOGONAL CURVILINEAR COORDINATES IN AN EUCLIDEAN SPACE

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Summary

In this paper a non-iterative method for the numerical generation of orthogonal curvilinear coordinates for plane annular regions between two arbitrary smooth closed curves has been developed. The basic generating equation is the Gaussian equation for an Euclidean space which has been solved analytically. The method has been applied in many cases and these test results demonstrate that the proposed method can be readily applied to a wide variety of problems.

Introduction

The problem of generating orthogonal or non-orthogonal curvilinear coordinate systems in arbitrary domains is a problem of current interest in many branches of physics and engineering, and particularly in fluid mechanics and aerodynamics. The idea of generating coordinate meshes by numerically solving a set of partial differential equations under the boundary-geometric data as the boundary conditions arose with the work of Winslow[1]. Later Barfield[2], Chu[3], Godunov and Prokopov[4], Amsden and Hirt[5] and Potter and Tuttle[6] used this concept in generating coordinate systems for particular physical situations. The whole concept has however been used in a much more organized manner by Thompson, Thames and Mastin[7] (later referred to as the TTM method) in developing and coding[8] the computer program for generating non-orthogonal coordinates in a variety of two-dimensional situations. The user, however, has no control over the orthogonality or non-orthogonality of the generated coordinates.

The underlying basis of all the above methods, including that of Pope[9], Starius[10], Middlecoff and Thomas[11], and Mobley and Stewart[12] is the choice of a set of coupled partial differential equations. Two exceptions to the above are the methods of Eiseman[13], whose method is of an algebraic-geometric nature, and that of Davis[14] which is based on the Schwarz-Christoffel transformation of the complex variable theory.

In the differential equations method, except for the work of Starius[10] where a hyperbolic system of equations is used, all other methods rest on the system of elliptic partial differential equations. The elliptic system is usually a set of Laplace or Poisson equations $\nabla^2\xi = -f_1$ and $\nabla^2\eta = -f_2$, where ξ = const. and η = const. are the coordinate curves with η = const. on all the boundary curves. Looking a little deeper, one finds

that these equations provide a set of differential constraints or relations among the fundamental metric coefficients g_{11}, g_{12} and g_{22}. The next step is to interchange the role of dependent and independent variables and then to solve the coupled system for the Cartesian coordinates x and y. In the TTM method, the arbitrariness of $f_1(\xi,\eta)$ and $f_2(\xi,\eta)$ has been used to control or redistribute the coordinate lines in the desired regions.

In this paper we develop a new approach based on providing another relationship among the fundamental metric coefficients which is not based on any arbitrary assumption. This relationship is provided by the condition that the coordinates are to be generated in an Euclidean space. The most natural choice is then to use the Gaussian equation[15] for an Euclidean space, viz., a space of zero curvature. This fundamental equation is one equation in the three unknowns g_{11}, g_{12} and g_{22}. To close this equation we can use the simplest elliptic system of two Poisson equations.

The preceding ideas have been tested in the generation of orthogonal coordinates in the annular region between two arbitrary smooth closed curves. In the case of orthogonal coordinates (i.e., $g_{12} = 0$) the resulting equations show that g_{22} is a function of ξ,η and g_{11} so that a single equation for the determination of $g_{11} = x_\xi^2 + y_\xi^2$ is obtained. A general solution of this equation with $g_{11} = g_{22}$ can be written down in a series form with the Fourier-coefficients determined from the prescribed values of g_{11} at the inner and outer boundaries. Further, from the earlier work of Potter and Tuttle[6] we have the result that in the case of orthogonal coordinates the ratio g_{11}/g_{22} is a product of functions of ξ and η. This result can be used to devise new coordinates ξ' and η' in which the resulting equation

is again of the same form as in ξ and η. Thus the same solution can be used with a change of variables to provide the solution when $g_{22} \neq g_{11}$, either with or without coordinate redistribution.

The method developed on the preceding ideas therefore provides a non-iterative closed form analytic solution for the case of two-dimensional orthogonal coordinates. The main methodology is detailed in the succeeding sections. Numerical results of the generated coordinates are shown in Figures 3-8.

Formulation of the Problem

For the purpose of continuity of presentation, we first state the following well known results: the line element ds in any space is given by the Riemannian formula

$$(ds)^2 = g_{ij}dx^i dx^j \tag{1}$$

where the g_{ij}'s are the covariant components of the fundamental metric tensor. The choice of the coordinate system (x^k) for any space is quite arbitrary, viz., any coordinate system can be introduced for reference purposes, however, the values of $g_{ij}(x^k)$ and their distributions "in the small" depend both on the coordinate system and on the intrinsic geometry of the space in such a manner that the same value of ds is obtained irrespective of the chosen coordinate system. Further, if in the chosen space it is possible to introduce a set of rectangular coordinate axes, then the line element is also given by

$$(ds)^2 = \delta_{ij}dx^i dx^j \tag{2}$$

where δ_{ij} is the Kronecker delta and (x^k) is now a rectangular Cartesian system. A space in which, in addition to any general coordinate system, the line element is also obtainable through equation (2), is known as an Euclidean space.

The preceding ideas can be condensed by introducing the concept of curvature of the chosen space in which the coordinate system (x^k) has been introduced. If the Riemannian curvature of a space is zero then the space is said to be Euclidean and it is then possible to introduce rectangular Cartesian coordinates in this space. Therefore, if a two-dimensional space is Euclidean then the Riemannian or Gaussian formula expressing the relation

among the g_{ij}'s for any coordinate system, (while writing $x^1 = \xi$, $x^2 = \eta$), is given by

$$\frac{\partial}{\partial \eta}\left(\frac{\sqrt{g}\,\Gamma_{11}^2}{g_{11}}\right) - \frac{\partial}{\partial \xi}\left(\frac{\sqrt{g}\,\Gamma_{12}^2}{g_{11}}\right) = 0 \tag{3}$$

where Γ_{jk}^i are the Christoffel symbols of the second kind defined by

$$\Gamma_{jk}^i = \frac{1}{2} g^{i\ell}\left(\frac{\partial g_{\ell j}}{\partial x^k} + \frac{\partial g_{\ell k}}{\partial x^j} - \frac{\partial g_{jk}}{\partial x^\ell}\right) \tag{4}$$

and

$$g = g_{11}\, g_{22} - (g_{12})^2.$$

The main theme of the present paper is the choice of the fundamental equation (3) for the determination of g_{ij}'s and then to determine the rectangular Cartesian coordinates x and y as functions of ξ and η.

Determination of x and y

Equation (3) implies that there exists a continuous function $\alpha(\xi,\eta)$ such that

$$\left.\begin{aligned} \alpha_\xi &= \frac{-\sqrt{g}}{g_{11}}\,\Gamma_{11}^2 \\ \alpha_\eta &= \frac{-\sqrt{g}}{g_{11}}\,\Gamma_{12}^2 \end{aligned}\right\} \tag{5}$$

where a variable subscript denotes partial differentiation. Based on the following formulae

$$\left.\begin{aligned} &g_{11} = x_\xi^2 + y_\xi^2,\ g_{12} = x_\xi x_\eta + y_\xi y_\eta,\ g_{22} = x_\eta^2 + y_\eta^2 \\ &\Gamma_{11}^2 = \frac{1}{\sqrt{g}}(x_\xi y_{\xi\xi} - y_\xi x_{\xi\xi}),\ \Gamma_{12}^2 = \frac{1}{\sqrt{g}}(x_\xi y_{\xi\eta} - y_\xi x_{\xi\eta}) \end{aligned}\right\} \tag{6}$$

it is easy to show by direct substitution that

$$\left.\begin{array}{c} x_\xi = \sqrt{g_{11}}\cos\alpha,\ y_\xi = -\sqrt{g_{11}}\sin\alpha \\ \\ x_\eta = \dfrac{1}{\sqrt{g_{11}}}(g_{12}\cos\alpha + \sqrt{g}\sin\alpha),\ y_\eta = \dfrac{1}{\sqrt{g_{11}}}(\sqrt{g}\cos\alpha - g_{12}\sin\alpha) \end{array}\right\} \quad (7)$$

Using equations (5) and (7) in the expressions for $d\alpha$, dx and dy, we obtain the expressions for α, x and y (first obtained by Martin[16]).

$$\alpha = -\int \frac{\sqrt{g}}{g_{11}}(\Gamma^2{}_{11}\, d\xi + \Gamma^2{}_{12}\, d\eta) \quad (8)$$

$$x = \int[\sqrt{g_{11}}\cos\alpha\, d\xi + \frac{1}{\sqrt{g_{11}}}(g_{12}\cos\alpha + \sqrt{g}\sin\alpha)d\eta] \quad (9)$$

$$y = \int[-\sqrt{g_{11}}\sin\alpha\, d\xi + \frac{1}{\sqrt{g_{11}}}(\sqrt{g}\cos\alpha - g_{12}\sin\alpha)d\eta] \quad (10)$$

The geometrical interpretation of α is that it is the angle of inclination with respect to the x-axis of the tangent to the coordinate line η = const. directed in the sense of increasing values of the parameter ξ. The choice of the minus sign in equation (5) is due to the adopted convention that both ξ and α be treated as positive in the clockwise sense.

Case of Orthogonal Coordinates

Equation (3) when written in terms of g_{ij} and $g = g_{11}g_{22} - (g_{12})^2$ has the form

$$\frac{\partial}{\partial\xi}\left(\frac{g_{12}}{g_{11}\sqrt{g}}\frac{\partial g_{11}}{\partial\eta} - \frac{1}{\sqrt{g}}\frac{\partial g_{22}}{\partial\xi}\right)$$

$$+ \frac{\partial}{\partial\eta}\left(\frac{2}{\sqrt{g}}\frac{\partial g_{12}}{\partial\xi} - \frac{1}{\sqrt{g}}\frac{\partial g_{11}}{\partial\eta} - \frac{g_{12}}{g_{11}\sqrt{g}}\frac{\partial g_{11}}{\partial\xi}\right) = 0 \quad (11)$$

which is an equation in three unknowns, viz., g_{11}, g_{12} and g_{22}. Here a wide range of possibilities are open to express g_{12} and g_{22} as functions of g_{11} either through algebraic or differential relations. Fortunately, in the case of orthogonal coordinates this arbitrariness is minimal and equation (11) can be reduced to a very simple form which can be solved analytically. The following discussion and analysis pertains only to orthogonal coordinates.

In the case of orthogonal coordinates, the coefficient g_{12} is zero, i.e.,

$$g_{12} = x_\xi x_\eta + y_\xi y_\eta = 0 \tag{12}$$

which is satisfied by the equations

$$\left.\begin{aligned} x_\eta &= -F\, y_\xi \\ y_\eta &= F\, x_\xi \end{aligned}\right\} \tag{13}$$

where $F(x,y,\xi,\eta,x_\xi,y_\xi,x_\eta,y_\eta) > 0$ is a continuous function of its arguments.

Let the boundary Γ_2 of a bounded region Ω in an Euclidean two-dimensional space be a simple smooth curve $x = x_\infty(\xi)$, $y = y_\infty(\xi)$, with a uniformly turning tangent. In the region Ω, let Ω_s be an annular subregion bounded by the inner boundary Γ_1 and the outer boundary Γ_2 as shown in Figure 1. The region Ω_s is to be mapped onto a rectangular region R in the $\xi\eta$-plane by a transformation with the initial data

$$\left.\begin{aligned} x(\xi,\eta_\infty) &= x_\infty(\xi) \\ y(\xi,\eta_\infty) &= y_\infty(\xi) \end{aligned}\right\} 0 \le \xi \le \xi_m \tag{14}$$

so as to have

$$\left.\begin{aligned} x &= x(\xi,\eta) \\ y &= y(\xi,\eta) \end{aligned}\right\} \eta_\beta \le \eta \le \eta_\infty \tag{15}$$

where η_β and η_∞ are the actual parametric values associated with the boundaries Γ_1 and Γ_2, respectively, and x and y are periodic in the ξ-argument with the period

$$2\ell = \xi_m \tag{16}$$

The set of equations (13) along with the initial data (14) can form a well-posed initial-value problem for hyperbolic equations if certain conditions on F are satisfied. It is to be noted that in this case no boundary data is needed on the curve Γ_1. This problem has been considered by Starius[10].

Since in this paper the equation for the condition of an Euclidean space (equation (11)) forms the basis of the proposed method, we expect to have an elliptic boundary-value problem to be solved under the Dirichlet conditions. Equation (11) with the substitution $g_{12} = 0$ and on using equations (13) becomes

$$\frac{\partial}{\partial\xi}\left[\frac{1}{Fg_{11}}\frac{\partial}{\partial\xi}(F^2 g_{11})\right] + \frac{\partial}{\partial\eta}\left[\frac{1}{Fg_{11}}\frac{\partial g_{11}}{\partial\eta}\right] = 0 \tag{17}$$

$$g_{22} = F^2 g_{11}$$

$$g = (Fg_{11})^2 \tag{18}$$

Equation (17) can now be solved as an elliptic boundary-value problem in g_{11} provided that it can be proved that F is a function of ξ,η and g_{11}. These considerations on F will also provide a class of functions from which F can be chosen in a simple way.

In Reference 10, Starius has proved the following properties of F: (a) F is an invariant under translation and rotation of the coordinate axes (x,y). Therefore, if (x,y) is a solution of equations (13), then it

can be shown that

$$\hat{x} = a + x \cos\theta - y \sin\theta$$

$$\hat{y} = b + x \sin\theta + y \cos\theta$$

is a solution of the equations

$$\hat{x}_\eta = - F\, \hat{y}_\xi$$

$$\hat{y}_\eta = F\, \hat{x}_\xi$$

where a,b and θ are arbitrary constants. These considerations show that F is not an explicit function of x and y.

(b) On the basis of the results obtained in (a), we have

$$F(\xi,\eta,\ \hat{x}_\xi,\ \hat{y}_\xi,\ \hat{x}_\eta,\ \hat{y}_\eta) = F(\xi,\eta,\ x_\xi,\ y_\xi,\ x_\eta,\ y_\eta)$$

hence $\frac{dF}{d\theta} = 0$ for all values of θ including $\theta = 0$. Evaluating $\left(\frac{dF}{d\theta}\right)_{\theta = 0}$, we obtain

$$-y_\xi \frac{\partial F}{\partial x_\xi} + x_\xi \frac{\partial F}{\partial y_\xi} - y_\eta \frac{\partial F}{\partial x_\eta} + x_\eta \frac{\partial F}{\partial y_\eta} = 0$$

This equation shows that $F = F(\xi,\eta,\ g_{11},\ g_{22})$, so that F depends on x_ξ, x_η, y_ξ, y_η through g_{11} and g_{22} only.

(c) The considerations in (a) and (b) along with equation (18) show that

$$g_{22} = g_{11}\, F^2(\xi,\eta,\ g_{11},\ g_{22})$$

so that in principle g_{22} can be expressed as a function of g_{11} and consequently $F = F(\xi,\eta,\ g_{11})$. This proves the contention that in the case of orthogonal coordinates, equation (17) is sufficient for the calculation of g_{11}.

The set of functions $F > 0$ satisfying the conditions (a), (b), and (c) enumerated above also contain $F = 1$ as an element. This choice of F*

*This is by no means a restriction, as is demonstrated later.

yields the simplest possible form of the generating equation. With F = 1, equation (17) becomes

$$\frac{\partial^2 P}{\partial \xi^2} + \frac{\partial^2 P}{\partial \eta^2} = 0 \tag{19}$$

where

$$P = \ln g_{11},\ g_{22} = g_{11},\ \nabla^2 \xi = 0^{**} \tag{20}$$

The boundary conditions are

$$\left.\begin{aligned} P &= P_\beta(\xi) \quad \text{at} \quad \eta = 0^\dagger \\ &= P_\infty(\xi) \quad \text{at} \quad \eta = \eta_\infty \end{aligned}\right\} 0 \le \xi \le \xi_m \tag{21}$$

where the subscripts β and ∞ denote the inner and outer boundaries respectively. The periodicity requirement is that

$$P(\xi,\eta) = P(\xi + 2\ell,\ \eta) \tag{22}$$

where

$$2\ell = \xi_m \ .$$

A general analytic solution of equation (19) under the conditions (21) and (22) is

$$P(\xi,\eta) = a_o + \eta\bar{K} + \sum_{n=1}^{\infty} \sinh \frac{n\pi}{\ell}(\eta_\infty - \eta)\left(a_n \cos \frac{n\pi\xi}{\ell} + b_n \sin \frac{n\pi\xi}{\ell}\right)\Big/\sinh \frac{n\pi\eta_\infty}{\ell}$$

$$+ \sum_{n=1}^{\infty} \sinh \frac{n\pi\eta}{\ell}\left(c_n \cos \frac{n\pi\xi}{\ell} + d_n \sin \frac{n\pi\xi}{\ell}\right)\Big/\sinh \frac{n\pi\eta_\infty}{\ell} \tag{23}$$

where

$$\bar{K} = (c_o - a_o)/\eta_\infty \tag{24}$$

---

†There is no loss of generality in setting the parametric value $\eta_\beta = 0$. The value η_∞ must be interpreted as the difference between the actual values at the outer and inner boundaries. The determination of η_∞ is of crucial importance to this work and is discussed in the next section.

**Refer to the next section.

and

$$\left.\begin{aligned}
a_o &= \frac{1}{2\ell}\int_o^{2\ell} P_\beta(\xi)d\xi, \quad c_o = \frac{1}{2\ell}\int_o^{2\ell} P_\infty(\xi)d\xi \\
a_n &= \frac{1}{\ell}\int_o^{2\ell} P_\beta(\xi)\cos\frac{n\pi\xi}{\ell}\,d\xi, \quad b_n = \frac{1}{\ell}\int_o^{2\ell} P_\beta(\xi)\sin\frac{n\pi\xi}{\ell}\,d\xi \\
c_n &= \frac{1}{\ell}\int_o^{2\ell} P_\infty(\xi)\cos\frac{n\pi\xi}{\ell}\,d\xi, \quad d_n = \frac{1}{\ell}\int_o^{2\ell} P_\infty(\xi)\sin\frac{n\pi\xi}{\ell}\,d\xi
\end{aligned}\right\} \quad (25)$$

Having determined the coefficients a_n, b_n, c_n and d_n as defined in (25), we can obtain the values of g_{11} from (23) for all values of ξ and η. To find the expression for α we consider equations (5), which for orthogonal coordinates are

$$\alpha_\xi = \frac{1}{2\sqrt{g}}\frac{\partial g_{11}}{\partial \eta}, \quad \alpha_\eta = -\frac{1}{2\sqrt{g}}\frac{\partial g_{22}}{\partial \xi}.$$

For $g_{22} = g_{11}$, these equations become

$$\alpha_\xi = \frac{1}{2}\frac{\partial P}{\partial \eta}, \quad \alpha_\eta = -\frac{1}{2}\frac{\partial P}{\partial \xi}, \quad (26)$$

and on integration yield the exact expression for α.

$$\begin{aligned}
\alpha(\xi,\eta) = \alpha(\xi,o) &+ \sum_{n=1}^{\infty} \frac{\cosh\frac{n\pi}{\ell}(\eta_\infty-\eta)}{2\sinh\frac{n\pi\eta_\infty}{\ell}}\left(b_n\cos\frac{n\pi\xi}{\ell} - a_n\sin\frac{n\pi\xi}{\ell}\right) \\
&+ \sum_{n=1}^{\infty} \frac{\cosh\frac{n\pi\eta}{\ell}}{2\sinh\frac{n\pi\eta_\infty}{\ell}}\left(c_n\sin\frac{n\pi\xi}{\ell} - d_n\cos\frac{n\pi\xi}{\ell}\right) \\
&- \sum_{n=1}^{\infty} \frac{\cosh\frac{n\pi\eta_\infty}{\ell}}{2\sinh\frac{n\pi\eta_\infty}{\ell}}\left(b_n\cos\frac{n\pi\xi}{\ell} - a_n\sin\frac{n\pi\xi}{\ell}\right) \\
&- \sum_{n=1}^{\infty} \frac{1}{2\sinh\frac{n\pi\eta_\infty}{\ell}}\left(c_n\sin\frac{n\pi\xi}{\ell} - d_n\cos\frac{n\pi\xi}{\ell}\right)
\end{aligned} \quad (27)$$

Since the line integrals for the determination of x and y (cf. Eqs. (9) and (10)) are independent of the path, viz.,

$$\frac{\partial}{\partial \eta}(\sqrt{g_{11}} \cos \alpha) = \frac{\partial}{\partial \xi}(\sqrt{g_{22}} \sin \alpha)$$

$$\frac{\partial}{\partial \eta}(\sqrt{g_{11}} \sin \alpha) = -\frac{\partial}{\partial \xi}(\sqrt{g_{22}} \cos \alpha)$$

hence

$$x(\xi,\eta) = x(\xi,o) + \int_o^\eta \sqrt{g_{22}} \sin \alpha \, d\eta \tag{28}$$

$$y(\xi,\eta) = y(\xi,o) + \int_o^\eta \sqrt{g_{22}} \cos \alpha \, d\eta \tag{29}$$

The preceding analysis completes the basic development of the subject.

Coordinate Re-Distribution (Contraction)

In order to have the capability of re-distributing the coordinate lines so as to have a control on the mesh spacings in the desired regions, we consider a transformation from (ξ,η) to new coordinates $(\bar{\xi},\bar{\eta})$. The transformation functions can arbitrarily be selected, but can also be linked in some manner to the physical field behaviors. For example, in viscous flow problems the effect of viscosity near a wall can be incorporated in the transformation functions. Below we proceed without specifying these functions and then give one example in the section on numerical method.

On transformation from (ξ,η) to $(\bar{\xi},\bar{\eta})$, the covariant metric coefficients transform as

$$\bar{g}_{ij} = g_{k\ell} \frac{\partial x^k}{\partial \bar{x}^i} \frac{\partial x^\ell}{\partial \bar{x}^j}$$

so that on using the relation $g_{22} = g_{11}$ and $g_{12} = 0$, we have

$$\left.\begin{aligned} \bar{g}_{11} &= \left[\left(\frac{\partial \xi}{\partial \bar{\xi}}\right)^2 + \left(\frac{\partial \eta}{\partial \bar{\xi}}\right)^2\right] g_{11} \\ \bar{g}_{22} &= \left[\left(\frac{\partial \xi}{\partial \bar{\eta}}\right)^2 + \left(\frac{\partial \eta}{\partial \bar{\eta}}\right)^2\right] g_{11} \end{aligned}\right\} \tag{30}$$

We now introduce the transformation

$$\left.\begin{aligned} \xi &= \phi(\bar{\xi}) \\ \eta &= f(\bar{\eta}) \end{aligned}\right\} \tag{31}$$

where the functions ϕ and f are continuously differentiable and satisfy the conditions

$$\phi(\bar{\xi}_o) = 0, \quad f(\bar{\eta}_\beta) = 0$$

where $\xi = 0$ and $\eta = 0$ correspond respectively to $\bar{\xi} = \bar{\xi}_o$ and $\bar{\eta} = \bar{\eta}_\beta$. Defining

$$\lambda = \frac{d\phi}{d\bar{\xi}}, \quad \theta = \frac{df}{d\bar{\eta}} \tag{31a}$$

we have from (30)

$$\bar{g}_{11}(\bar{\xi},\bar{\eta}) = \lambda^2 g_{11}(\bar{\xi},\bar{\eta}) \tag{32b}$$

$$\bar{g}_{22}(\bar{\xi},\bar{\eta}) = \theta^2 g_{11}(\bar{\xi},\bar{\eta}) \tag{32c}$$

$$= \frac{\theta^2}{\lambda^2}\, \bar{g}_{11}(\bar{\xi},\bar{\eta}) \tag{32d}$$

To obtain the solution in the $(\bar{\xi},\bar{\eta})$ coordinate system, we merely have to replace ξ and η by the functions $\phi(\bar{\xi})$ and $\phi(\bar{\eta})$ respectively in (23) and (27), while (28) and (29) become

$$X(\bar{\xi},\bar{\eta}) = X(\bar{\xi},\bar{\xi}_o) + \int_{\bar{\eta}_\beta}^{\bar{\eta}} \sqrt{\bar{g}_{22}}\, \sin\ \alpha(\bar{\xi},\bar{\eta})d\bar{\eta} \tag{33a}$$

$$Y(\bar{\xi},\bar{\eta}) = Y(\bar{\xi},\bar{\xi}_o) + \int_{\bar{\eta}_\beta}^{\bar{\eta}} \sqrt{\bar{g}_{22}}\, \cos\ \alpha(\bar{\xi},\bar{\eta})d\bar{\eta} \tag{33b}$$

It must be noted that on transformation the resulting metric coefficients $\bar{g}_{11}$ and $\bar{g}_{22}$ are not equal.

The salient feature of the preceding analysis is that the solution under the condition $g_{22} = g_{11}$ can be used to obtain the solution for the case $g_{22} \neq g_{11}$ by coordinate transformation.

Uniqueness Condition on ξ for Orthogonal Coordinates

Any method for the generation of orthogonal coordinates on the preceding lines has to be supplemented with a uniqueness condition on the behavior of ξ and a method for its selection. The following analysis, besides covering the above two aspects, also provides a general basis for the earlier choice F = 1.

For the case of orthogonal coordinates, equations (7) can also be expressed in the inverse form as

$$\left.\begin{aligned} \xi_x &= \cos\alpha/\sqrt{g_{11}}, \quad \xi_y = -\sin\alpha/\sqrt{g_{11}} \\ \eta_x &= \sin\alpha/\sqrt{g_{22}}, \quad \eta_y = \cos\alpha/\sqrt{g_{22}} \end{aligned}\right\}$$

Writing $F = \sqrt{g_{22}/g_{11}}$, eliminating α between the above equations and then by cross differentiation, we obtain the following equations:

$$\frac{\partial}{\partial x}(\xi_x/F) + \frac{\partial}{\partial y}(\xi_y/F) = 0 \tag{34}$$

$$\frac{\partial}{\partial x}(F\eta_x) + \frac{\partial}{\partial y}(F\eta_y) = 0 \tag{35}$$

Following Protter and Tuttle[6] we assume that the ξ-curves in the xy-plane are free from sources and sinks. This condition establishes a unique correspondence between the ξ points on each pair of η = constant lines. In the absence of sources or sinks, we have

$$\operatorname{div}[\operatorname{grad}\psi(\eta)] = 0 \tag{36}$$

where $\psi(\eta)$ is an arbitrary differentiable function of η, and as such grad $\psi(\eta)$ is oriented along the normal to the curve η = const. Using the expressions

$$|\operatorname{grad}\eta| = 1/\sqrt{g_{22}}, \quad \nabla^2\eta = \frac{1}{\sqrt{g}}\frac{\partial}{\partial\eta}\left(\sqrt{\frac{g_{11}}{g_{22}}}\right)$$

in (36), we obtain

$$\frac{\partial}{\partial\eta}\left(\ell n \sqrt{g_{11}/g_{22}}\right) = -\frac{d^2\psi}{d\eta^2} \Big/ \frac{d\psi}{d\eta}$$

Writing $\frac{d\psi}{d\eta} = 1/\nu(\eta)$ and denoting the arbitrary function due to integration as $\ell n\mu(\xi)$, we obtain the result

$$\sqrt{g_{11}/g_{22}} = \frac{1}{F} = \mu(\xi)\ \nu(\eta) \tag{37}$$

Introducing new variables

$$\xi' = \int\mu(\xi)d\xi, \quad \eta' = \int \frac{d\eta}{\nu(\eta)} \tag{38}$$

and using (37) in (34) and (35), we get

$$\nabla^2\xi' = 0 \tag{39}$$

$$\nabla^2\eta' = 0 \tag{40}$$

Further using (37) and (38) in the fundamental equation (17), we obtain

$$\frac{\partial^2 p'}{\partial\xi'^2} + \frac{\partial^2 p'}{\partial\eta'^2} = 0 \tag{41}$$

where

$$p' = \ell n\ g'_{11}$$

$$g'_{22} = g'_{11}$$

$$g'_{11} = x^2_{\xi'} + y^2_{\xi'}, \quad g'_{22} = x^2_{\eta'} + y^2_{\eta'} .$$

The solution of equation (41) is of the same form as that of equation (19), viz., (23), and is obtained by replacing ξ and η by ξ' and η'. However, the important result obtained here is that a generating equation of the form (19) or (41) must be supplemented with a Laplace equation for ξ or ξ' respectively. This is the result or the required uniqueness condition on ξ for the generation of orthogonal coodinates. The condition

on η', i.e., equation (40), is implicitly satisfied by the coupled system of equations (39) and (41) and needs no discussion.

The condition equation $\nabla^2\xi = 0$ can rigorously be satisfied in all cases if we take ξ as the angle traced out in a clockwise sense by the common radius of the concentric circles in a conformal representation of the inner and outer boundaries. The numerical scheme for this aspect of the problem is an iterative one. In place of this elaborate scheme we have devised another method which is much simpler and non-iterative. Both of these methods have been discussed in the next section.

Numerical Method of Solution

Based on the formulation of the problem as discussed in the preceeding sections, we now have a non-iterative algebraic computational problem which can be handled in a straight forward manner. However, before solving a specific problem, it is important first to establish an orthogonal correspondence between unique points of the inner and outer boundary curves which are to be connected by a specified number of ξ = constant curves, and second to obtain the numerical value of the parametric difference η_∞. Two methods for the establishment of $x(\xi)$ and $y(\xi)$ are given below.

Method 1: It has been mentioned in equation (20) and later discussed in the preceeding section that the curves ξ = constant in the physical xy-plane must satisfy the Laplace equation $\nabla^2\xi = 0$. For this condition to be satisfied we can take ξ as the angle traced out in the clockwise sense by the common radius of the concentric circles in a conformal representation of the inner and outer boundary curves as follows.

The function $z = f(z')$, which conformally maps the region of the z-plane exterior to the specified curve C onto the region of the z′-plane exterior to the circle C′ of radius a, can be represented by a Laurent's expansion as

$$z = z' + p_o + iq_o + \sum_{n=1}^{\infty} (p_n + iq_n)\left(\frac{a}{z'}\right)^n \qquad (42)$$

For points on the circumference of the circle C′

$$z' = a\, e^{-i\xi}$$

so that

$$x(\xi) = p_o + (p_1+a)\cos\xi - q_1 \sin\xi + \sum_{n=2}^{\infty} (p_n \cos n\xi - q_n \sin n\xi) \quad (43)$$

$$y(\xi) = q_o + (p_1-a)\sin\xi + q_1 \cos\xi + \sum_{n=2}^{\infty} (p_n \sin n\xi + q_n \cos n\xi) \quad (44)$$

The same form of equations can be written for the outer boundary with A as the radius of the circle in the conformal plane.

Now $y = y(x)$ is known either in functional or tabular form. Thus starting from an initial guess for x the corresponding ordinates are used to determine the Fourier coefficients from equation (44) which in turn determine a new set of abscissae and then the ordinates, and so on. The convergence of this iterative method yields $x = x(\xi)$ and $y = y(\xi)$ both for the inner and outer boundaries. Note that after the completion of convergence we have

$$a = \frac{1}{2\pi}\int_o^{2\pi} [x_\beta(\xi)\cos\xi - y_\beta(\xi)\sin\xi]d\xi \quad (45a)$$

and

$$A = \frac{1}{2\pi}\int_o^{2\pi} [x_\infty(\xi)\cos\xi - y_\infty(\xi)\sin\xi]d\xi \quad (45b)$$

Method II: In lieu of using Method I, we have obtained equally good results by proceeding as follows. This method looks to be equivalent to Method I.

The inner and outer boundary data is available to us either in tabular or functional form as

$$y_\beta = y(x_\beta),\ y_\infty = y(x_\infty) \quad (46)$$

We now circumscribe circles around the inner and outer boundary curves. Two cases arise depending on whether the circles are concentric or nonconcentric.

Case I: If the circumscribed circles are concentric (Fig. 2a), then we select those sets of ordinates which correspond to the abscissae

$x_\beta = r_s \cos\xi$ and $x_\infty = r_L \cos\xi$, where r_s and r_L are the radii of the circumscribed circles.

Case II: If the circumscribed circles are non-concentric (Fig. 2b), then we first use the formula for the conformal transformation of non-concentric to concentric circles (Kober)[17] and choose the abscissae by using the following equation.

$$x(\xi) = [(1 - c\gamma\cos\xi)\{x_L (1 - c\gamma\cos\xi) + c\gamma\, y_L \sin\xi$$

$$+ r_L(c\cos\psi - \gamma\cos(\xi - \psi))\}$$

$$- c\gamma\sin\xi\{y_L(1 - c\gamma\cos\xi) - c\gamma\, x_L \sin\xi$$

$$- r_L(c\sin\psi + \gamma\sin(\xi - \psi))\}]$$

$$/(1 - 2c\gamma\cos\xi + c^2\gamma^2) \tag{47}$$

where

r_L, r_s = radii of outer and inner circumscribed circles.

(x_L, y_L) and (x_s, y_s) = coordinates of the centers.

$$d^2 = (x_s - x_L)^2 + (y_s - y_L)^2$$

$$\psi = \pi - \tan^{-1}\left(\frac{y_s - y_L}{x_s - x_L}\right)$$

$$c = [(d^2 + r_L^2 - r_s^2) + \{(d^2 + r_L^2 - r_s^2)^2 - 4d^2 r_L^2\}^{1/2}]/2dr_L$$

$$t = cr_L$$

$\gamma = 1$ for the outer boundary.

$$\gamma = \frac{r_L}{r_s}\left|\frac{d - t}{t}\right| \quad \text{for the inner boundary}$$

The ordinates are now selected corresponding to the set of abscissae given by (47).

Determination of η_∞

The parametric difference η_∞ is connected in some manner with the "modulus" of the domain. Determination of the modulus for annular regions has been considered by Burbea[18] and Gaier[19]. In this paper we base the determination of η_∞ on the radii a and A in the conformal representation process described in equations (45), and define it as

$$\eta_\infty = \ell n(\frac{A}{a}) \tag{48}$$

If Method I is used then A and a are available as parts of the converged iteration, while if Method II is used, then they are obtained by simple quadratures applied to equation (45).

Having determined the appropriate $x(\xi)$ and $y(\xi)$ both for the inner and outer boundaries, we first calculate the values of $(g_{11})_\beta$ and $(g_{11})_\infty$ numerically and then of $P_\beta(\xi)$ and $P_\infty(\xi)$. Based on these distributions the Fourier coefficients a_n, b_n, c_n and d_n are computed by numerical quadrature through the use of equation (25). Since these values of the coefficients are independent of the spacings between η = *constant* lines the same values are used when a redistribution of η = constant lines is desired. Substituting the Fourier coefficients in equation (23) we determine the values of $P(\xi,\eta)$ and hence of $g_{11}(\xi,\eta)$ for the whole annular region. A knowledge of $P(\xi,\eta)$ determines $\alpha(\xi,\eta)$ through equation (27) and so also the Cartesian coordinates $x(\xi,\eta)$ and $y(\xi,\eta)$ through equations (28) and (29) by numerical quadrature.

A computer program with the option of redistributing the coordinate lines in any desired manner has been written and used to generate the orthogonal coordinates for various annular regions as shown in Figures 3-8.

For the example problems we have chosen the following forms of the function ϕ and f of equations (31).

$$\left.\begin{aligned} \phi(\bar{\xi}) &= \frac{2\ell(\bar{\xi}-\bar{\xi}_o)}{\bar{\xi}_m-\bar{\xi}_o} \\ f(\bar{\eta}) &= \frac{\eta_\infty(\bar{\eta}-\bar{\eta}_\beta)}{\bar{\eta}_\infty-\bar{\eta}_\beta}\,\frac{K^{(\bar{\eta}-\bar{\eta}_\beta)}}{K^{(\bar{\eta}_\infty-\bar{\eta}_\beta)}} \end{aligned}\right\} \tag{49}$$

so that

$$\lambda = \frac{2\ell}{\bar{\xi}_m-\bar{\xi}_o}\,, \quad \theta = \frac{\eta_\infty}{\bar{\eta}_\infty-\bar{\eta}_\beta}\,[1+(\bar{\eta}-\bar{\eta}_\beta)\ell nK]\,\frac{K^{(\bar{\eta}-\bar{\eta}_\beta)}}{K^{(\bar{\eta}_\infty-\bar{\eta}_\beta)}} \tag{50}$$

where $K > 1$ is an arbitrary constant, $\ell = \pi$, and $\bar{\xi} = \bar{\xi}_m$, $\bar{\eta} = \bar{\eta}_\infty$ correspond respectively to $\xi = 2\ell = 2\pi$ and $\eta = \eta_\infty$. We treat $\bar{\xi}$ and $\bar{\eta}$ as integers so that $\bar{\xi}_o = 1$, $\bar{\xi}_m$ = IMAX, $\bar{\eta}_\beta = 1$, $\bar{\eta}_\infty$ = JMAX. Since η_∞ is known from (48), hence by specifying the numerical values to K and JMAX we can create the desired mesh control in the direction of η. The value of K between 1.05 and 1.1 is quite sufficient[20] to have very fine grid spacing near the inner boundary.

The number of terms to be retained in the series (23) is usually small for convex inner and outer boundary curves, though we have retained (IMAX-1)/2 number of coefficients in each computation. This number is the optimum number of terms in a discrete Fourier series[21] having IMAX number of points in one period. The average computer time for the complete computation on the UNIVAC 1100/80 for IMAX = 73 and JMAX = 60 field is about 2.75 minutes.

Summary of Numerical Experimentations

In the course of this investigation a number of cases of inner and outer boundary shapes and orientations have been tested through the developed computer program. The main conclusions are listed below:

(i) The method works very effectively for smooth and convex boundaries of any shape and orientation (cf. Figures (3)-(5) and (7)).

(ii) For concave boundaries a method similar to that of Eiseman[13] has to be used in the placement of the outer boundary to avoid intersecting ξ = constant lines (cf. Fig. 8).

(iii) Sharp turns or corners are not admissible and have to be rounded to avoid singularities (cf. Fig. 6).

Comparison with the TTM Method

As discussed in the introduction, the choice of the proposed method has been motivated by two considerations: (i) to choose a set of equations which are fundamental to the intrinsic nature of the space in which the coordinates are sought, and (ii) to minimize the inherent arbitrariness in the selection of relations among the metric coefficients. It is the purpose of this section to show that for the case of orthogonal coordinates the equations of TTM (Ref. 7,20) can be reduced in a simple form which can also be used to achieve the above two stated purposes.

The two-dimensional Laplacian of a scalar ψ of general coordinates ξ and η is

$$\nabla^2\psi = \frac{1}{\sqrt{g}}\left[\frac{\partial}{\partial\xi}\left(\frac{g_{22}\psi_\xi - g_{12}\psi_\eta}{\sqrt{g}}\right) + \frac{\partial}{\partial\eta}\left(\frac{g_{11}\psi_\eta - g_{12}\psi_\xi}{\sqrt{g}}\right)\right] \tag{51}$$

Introducing the operator

$$D^2 = g_{22}\partial_{\xi\xi} - 2g_{12}\partial_{\xi\eta} + g_{11}\partial_{\eta\eta} \tag{52}$$

in (51) gives another form

$$\nabla^2\psi = \frac{1}{g}D^2\psi + \psi_\xi\nabla^2\xi + \psi_\eta\nabla^2\eta \tag{53}$$

Writing $\psi = x$ and then $\psi = y$ in (53), we recover the equation of TTM, which are

$$\left.\begin{aligned} D^2x &= -g(x_\xi\nabla^2\xi + x_\eta\nabla^2\eta) \\ D^2y &= -g(y_\xi\nabla^2\xi + y_\eta\nabla^2\eta) \end{aligned}\right\} \tag{54}$$

For orthogonal coordinates $g_{12} = 0$, so that

$$\left.\begin{aligned} g_{22}x_{\xi\xi} + g_{11}x_{\eta\eta} &= -g(x_\xi\nabla^2\xi + x_\eta\nabla^2\eta) \\ g_{22}y_{\xi\xi} + g_{11}y_{\eta\eta} &= -g(y_\xi\nabla^2\xi + y_\eta\nabla^2\eta) \end{aligned}\right\} \tag{55}$$

Also, for orthogonal coordinates, equation (51) gives

$$\left.\begin{aligned} \nabla^2\xi &= \frac{1}{\sqrt{g_{11}g_{22}}} \frac{\partial}{\partial\xi} \left(\sqrt{g_{22}/g_{11}}\right) \\ \nabla^2\eta &= \frac{1}{\sqrt{g_{11}g_{22}}} \frac{\partial}{\partial\eta} \left(\sqrt{g_{11}/g_{22}}\right) \end{aligned}\right\} \tag{56}$$

It is immediately concluded from equations (56), that if the generating system is taken to be

$$\nabla^2\xi = 0, \ \nabla^2\eta = 0$$

then $g_{11}/g_{22} = K =$ constant. The choice $K = 1$ gives the condition for the Cauchy-Riemann equations. A change of variable such as $\eta' = \eta/\sqrt{K}$ achieves the same purpose. In any event, without loss of generality, equations (55) take the form

$$\left.\begin{aligned} x_{\xi\xi} + x_{\eta\eta} &= 0 \\ y_{\xi\xi} + y_{\eta\eta} &= 0 \end{aligned}\right\} \tag{57}$$

It will now be shown that even when the generating system is taken as

$$\nabla^2\xi = -f_1(\xi,\eta), \ \nabla^2\eta = -f_2(\xi,\eta),$$

f_1 and f_2 being arbitrary functions, the equations of the form (57) are again obtained.

Following the results of previous section, that, in the absence of sources or sinks in either the ξ or η lines, the most general form of g_{11}/g_{22} is

$$\sqrt{g_{11}/g_{22}} = \mu(\xi)\nu(\eta)$$

where μ and ν are arbitrary differentiable functions. Using (56) in (55) and defining new variables

$$\xi' = \int \mu(\xi)d\xi, \ \eta' = \int \frac{d\eta}{\nu(\eta)} \tag{58}$$

we obtain

$$\left.\begin{array}{l} x_{\xi'\xi'} + x_{\eta'\eta'} = 0 \\ y_{\xi'\xi'} + y_{\eta'\eta'} = 0 \\ \text{where} \\ \nabla^2\xi' = 0,\ \nabla^2\eta' = 0 \end{array}\right\} \tag{59}$$

From the foregoing analysis we conclude that the set of equations (59) are quite general and capable of generating orthogonal coordinates (as has earlier been proposed by Pope[9]), and there is no need to solve a more difficult and time consuming set of non-linear equations (55). Further the method of solving equations (59) follows the same patterns as that of the proposed method of this paper as discussed before.

Conclusions

A new method for the generation of orthogonal coordinates in two-dimensional regions has been proposed. The method shows that much of the arbitrariness in the choice of relations among the metrical coefficients can be minimized by the use of the condition of an Euclidean space. The method can also be used for simply connected regions only by obtaining the solution of the linear equation (19) under the changed boundary *conditions*. Besides, the proposed method can also be extended to three-dimensional regions.

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AFOSR 80-0185

References

1. A. J. Winslow, "Numerical Solution of the Quasi-Linear Equation in a Non-Uniform Triangular Mesh", Journal of Computational Physics, 2, 149 (1966).

2. W. D. Barfield, "An Optimal Mesh Generator for Lagrangian Hydrodynamic Calculations in Two Space Dimensions", Journal of Computational Physics, 6, 417 (1970).

3. W. H. Chu, "Development of a General Finite Difference Approximation for a General Domain, Part I: Machine Transformation", Journal of Computational Physics, 8, 392 (1971).

4. S. K. Godunov and G. P. Prokopov, "The Use of Moving Meshes in Gas Dynamics Computations", USSR Computational Mathematics and Mathematical Physics, 12, 182 (1972).

5. A. A. Amsden and C. W. Hirt, "A Simple Scheme for Generating General Curvilinear Grids", Journal of Computational Physics, 11, 348 (1973).

6. D. E. Potter and G. H. Tuttle, "The Construction of Discrete Orthogonal Coordinates", Journal of Computational Physics, 13, 483 (1973).

7. J. F. Thompson, F. C. Thames, and C. W. Mastin, "Automatic Numerical Generation of Body-Fitted Curvilinear Coordinate System for Field Containing any Number of Arbitrary Two-Dimensional Bodies", Journal of Computational Physics, 15, 299 (1974).

8. J. F. Thompson, F. C. Thames, and C. W. Mastin, "'TOMCAT'-A Code for Numerical Generation of Boundary-Fitted Curvilinear Coordinate System on Fields Containing any Number of Arbitrary Two-Dimensional Bodies", Journal of Computational Physics, 24, 274 (1977).

9. S. B. Pope, "The Calculation of Turbulent Recirculating Flows in General Orthogonal Coordinates", Journal of Computational Physics, 26, 197 (1978).

10. G. Starius, "Constructing Orthogonal Curvilinear Meshes by Solving Initial-Value Problem", Numerische Mathematik, 28, 25 (1977).

11. J. F. Middlecoff and P. D. Thomas, "Direct Control of the Grid Point Distribution in Meshes Generated by Elliptic Equations", AIAA Computational Fluid Dynamics Conference, Paper No. 79-1462 (1979).

12. C. D. Mobley and R. J. Stewart, "On the Numerical Generation of Boundary-Fitted Orthogonal Curvilinear Coordinate Systems", Journal of Computational Physics, 34, 124 (1980).

13. P. R. Eiseman, "A Coordinate System for a Viscous Transonic Cascade Analysis", Journal of Computational Physics, 26, 307 (1978).

14. R. T. Davis, "Numerical Methods for Coordinate Generation Based on Schwarz-Christoffel Transformation", AIAA Computational Fluid Dynamics Conference, Paper No. 79-1463 (1979).

15. L. P. Eisenhart, "Riemannian Geometry", Princeton University Press (1926).

16. M. H. Martin, "The Flow of a Viscous Fluid, I", Archiv of Rational Mechanics and Analysis, 41, 266 (1971).

17. H. Kober, "Dictionary of Conformal Representations", Dover Publications, Inc. (1952).

18. J. Burbea, "A Numerical Determination of the Modulus of Doubly Connected Domains by Using the Bergman Curvature", Mathematics of Computation, 25, 743 (1971).

19. D. Gaier, "Determination of Conformal Modulus of Ring Domains and Quadrilaterals", In-Lecture Notes in Mathematics, No. 399, Springer-Verlag, Berlin (1974).

20. Z. U. A. Warsi and J. F. Thompson, "Machine Solutions of Partial Differential Equations in the Numerically Generated Coordinate Systems", Engineering and Experimental Research Station, Mississippi State University, Rep. No. MSSU-EIRS-ASE-77-1, August (1976).

21. F. B. Hildebrand, "Introduction to Numerical Analysis", McGraw-Hill Book Company, Inc., (1956).

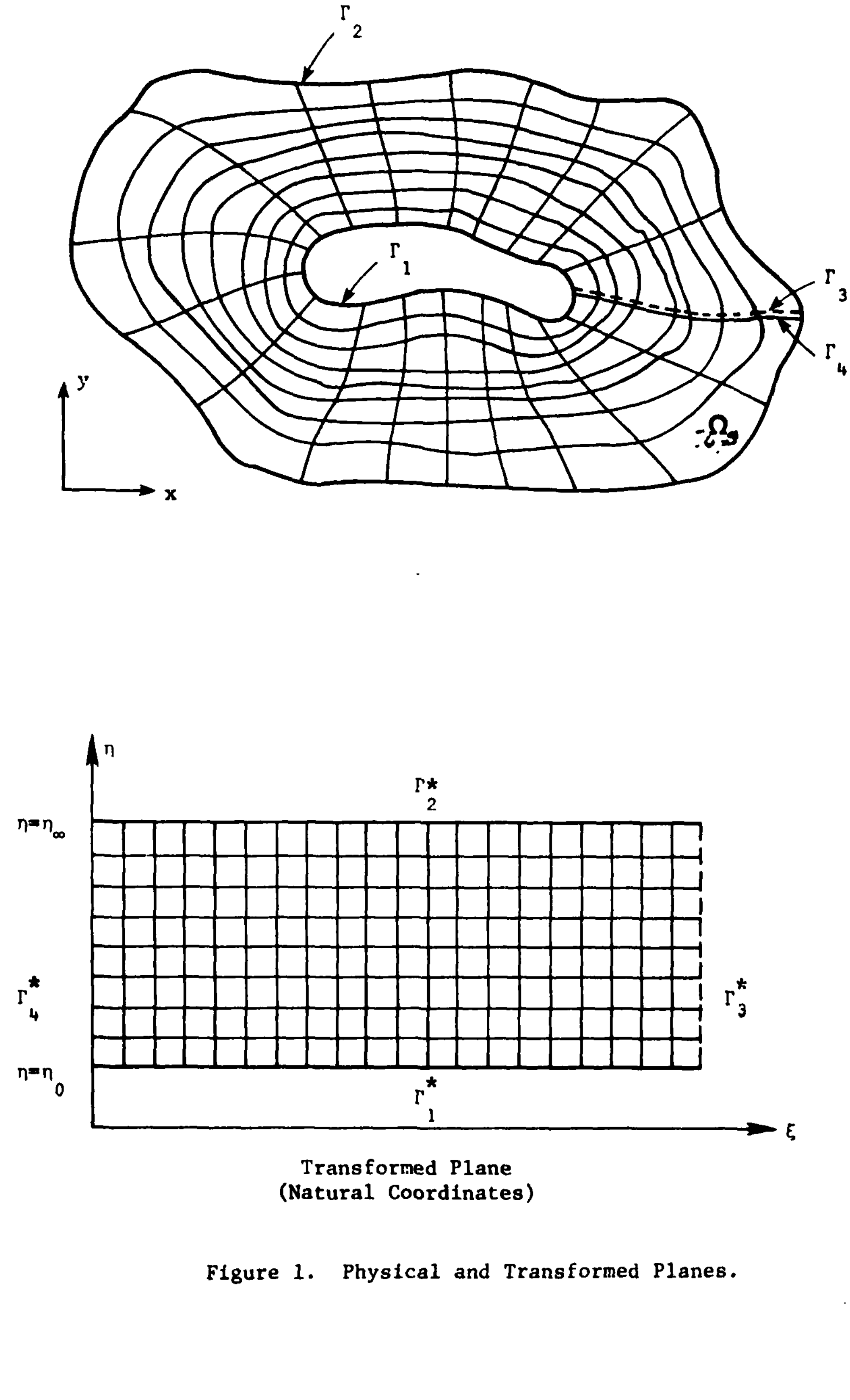

Figure 1. Physical and Transformed Planes.

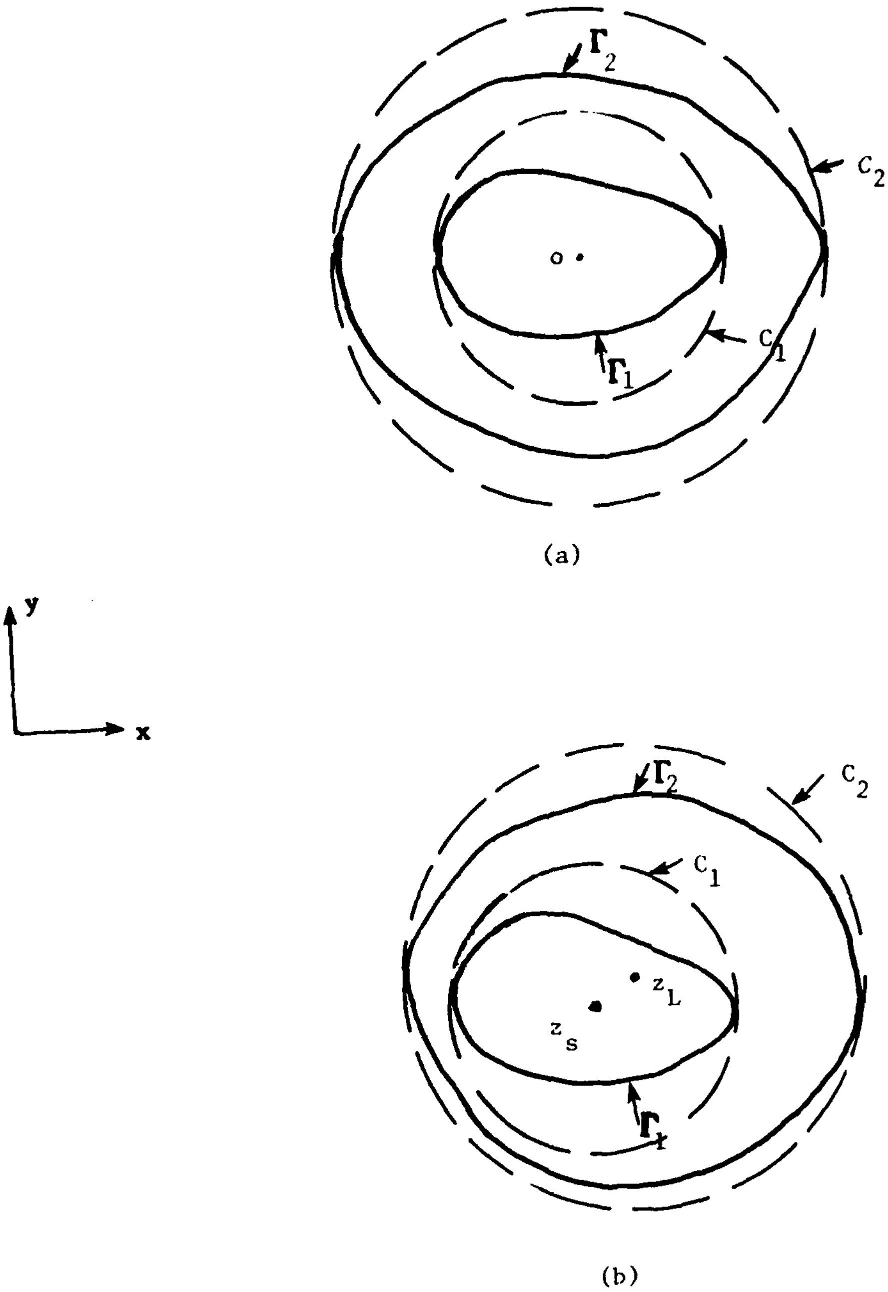

Figure 2. (a) Concentric circumscribed circles C_1 and C_2 of radii r_s, r_L respectively with center at the origin. (b) Non-concentric circumscribed circles C_1 and C_2 of radii r_s and r_L and centers at z_s and z_L respectively.

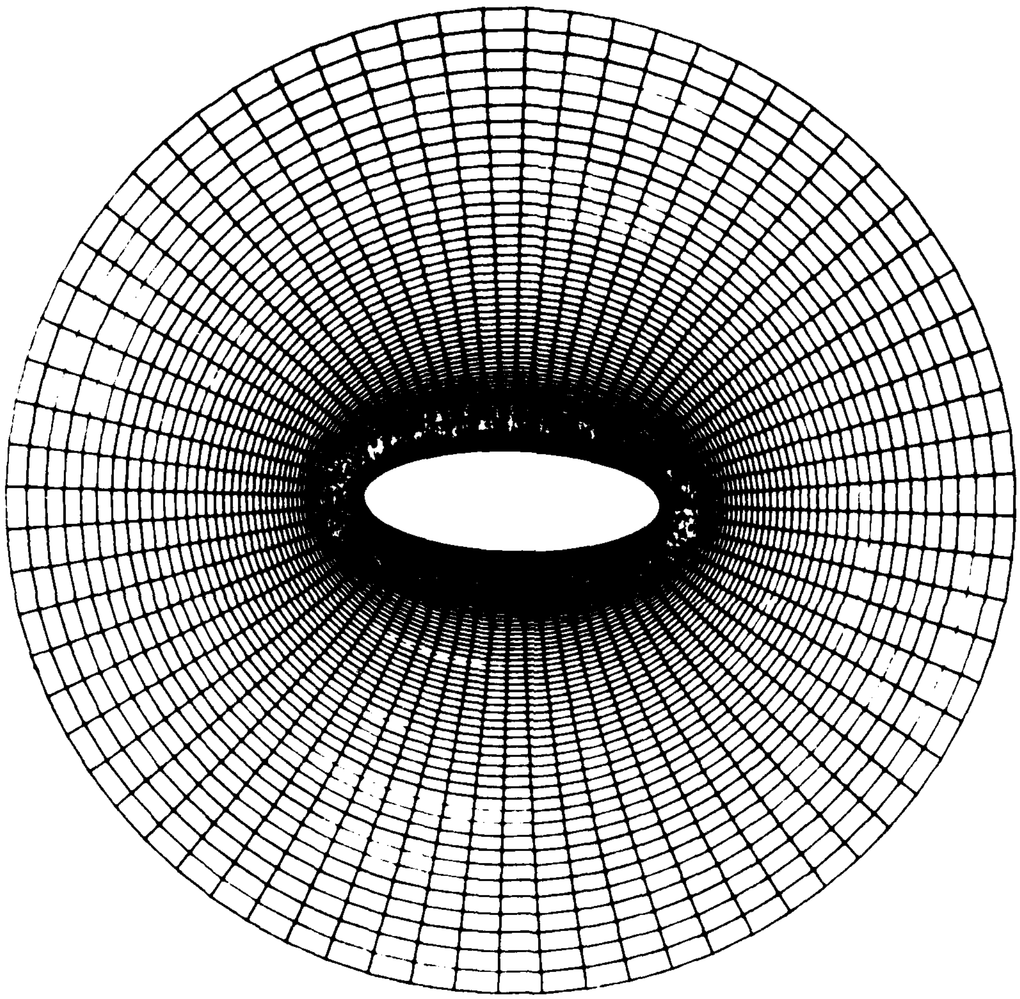

Figure 3. Confocal ellipses. Semi-major axes 1.48, 5.0, and semi-minor axes 0.5, 4.802 respectively. Only 38 η = const. lines shown for detail.

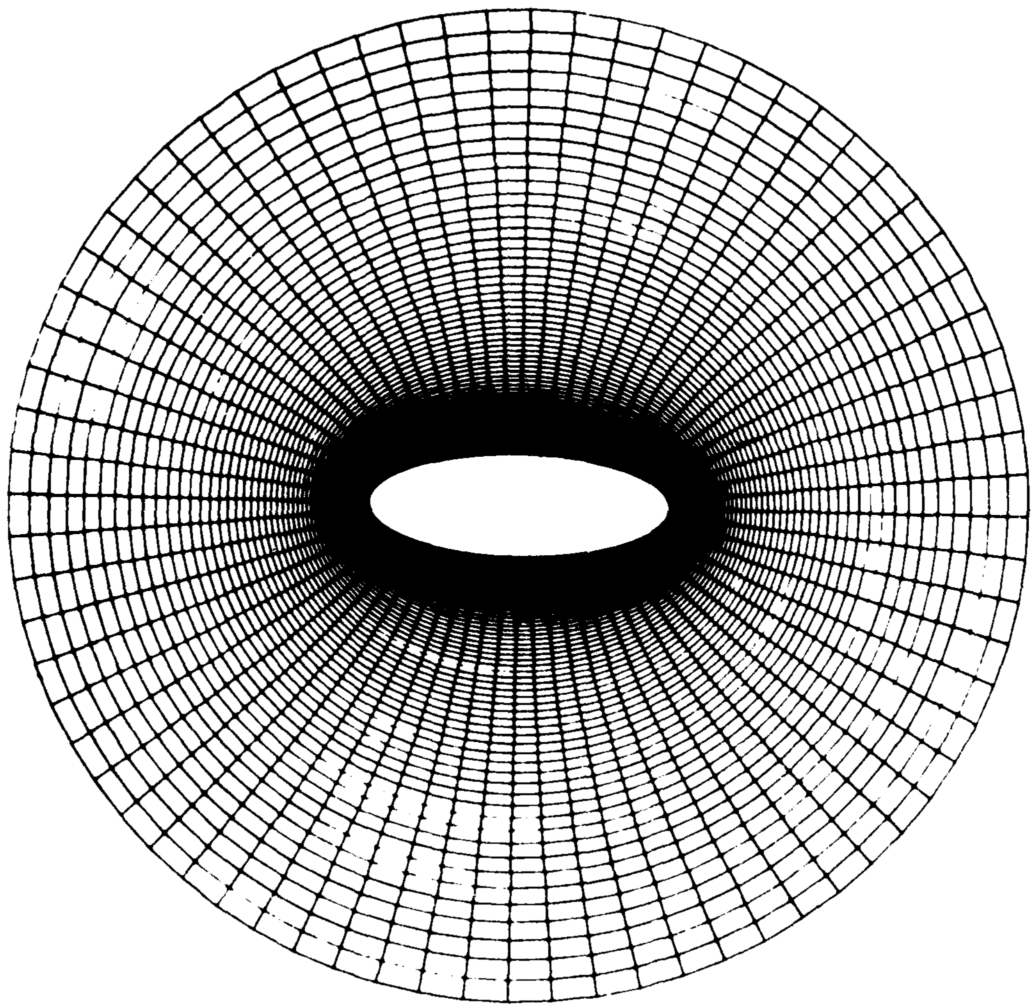

Figure 3. Confocal ellipses. Semi-major axes 1.48, 5.0, and semi-minor axes 0.5, 4.802 respectively. Only 38 η = const. lines shown for detail.

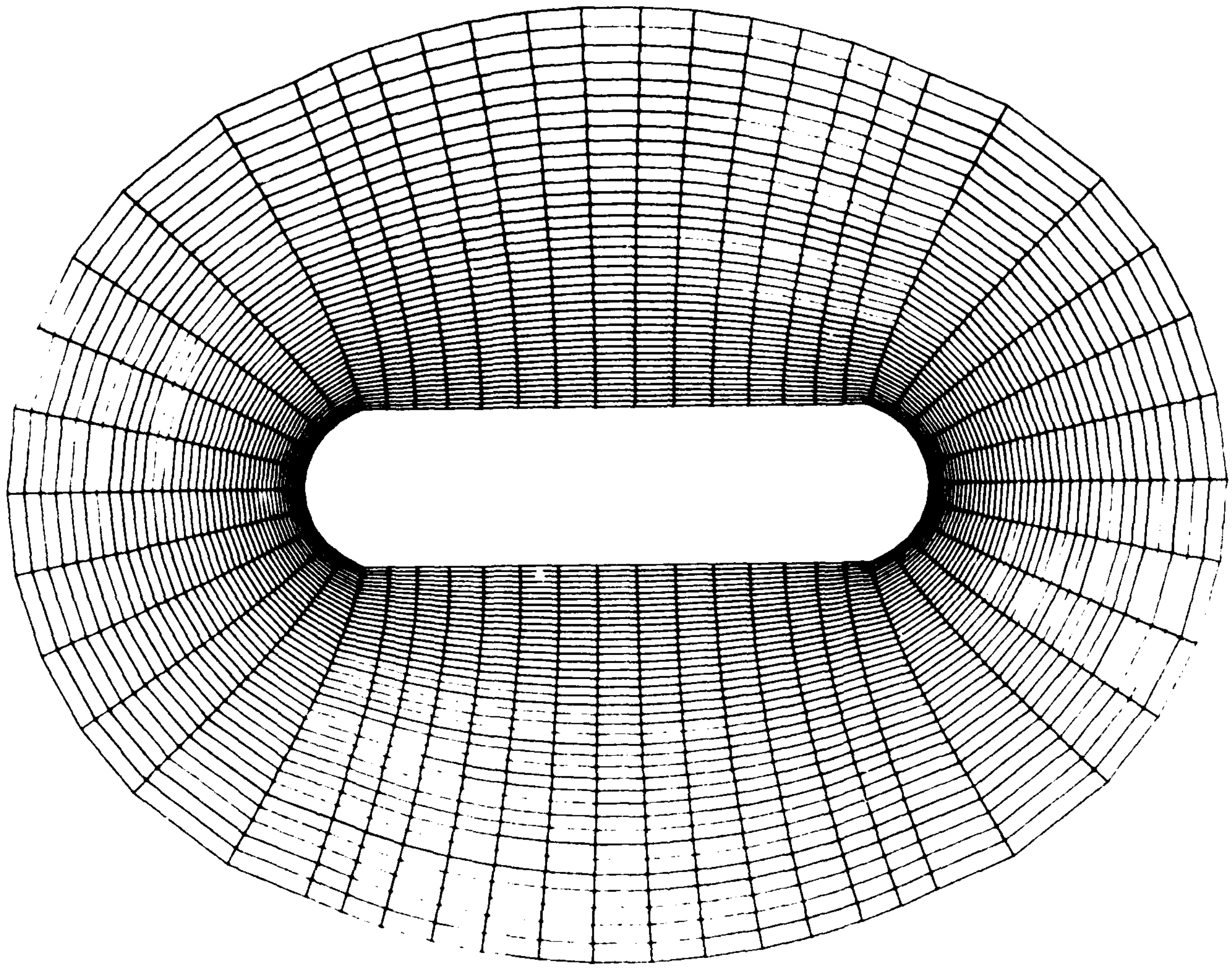

Figure 4. A blunt body section with elliptical outer boundary.

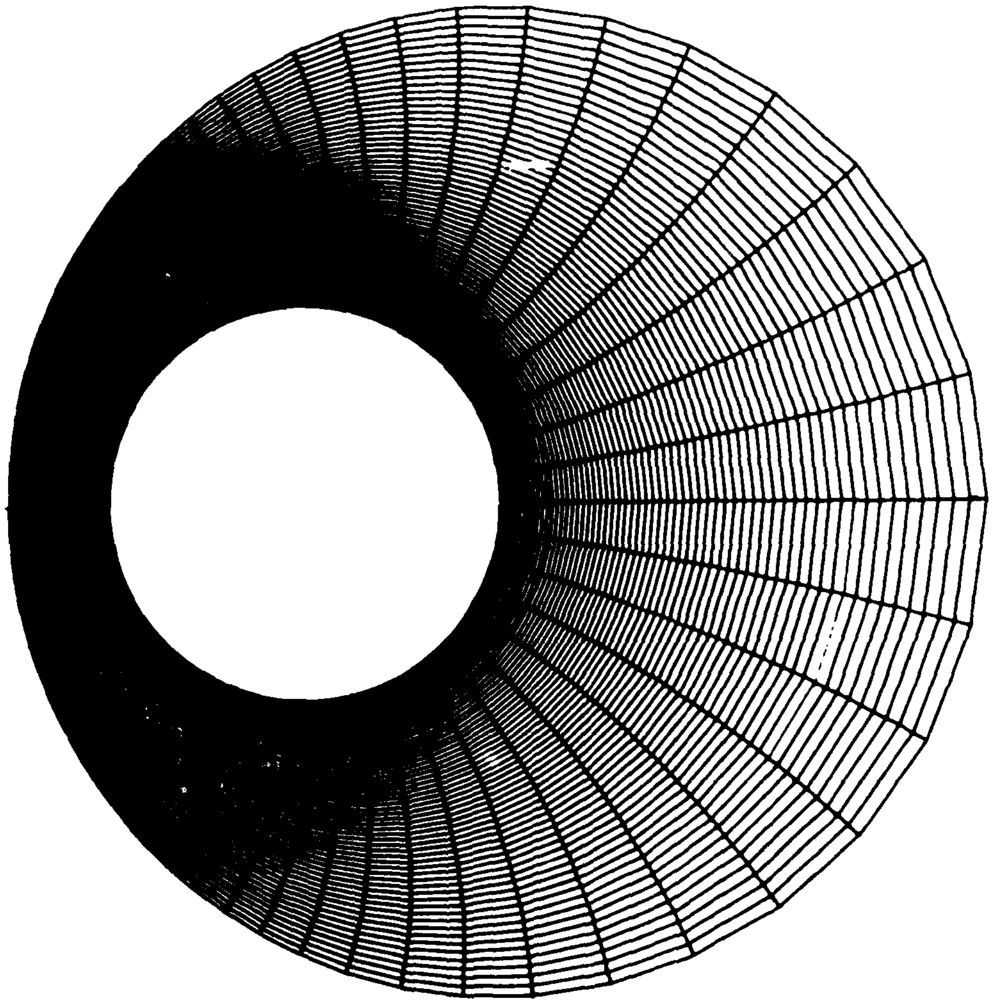

Figure 5. Non-concentric circles: $r_s = 1$, $r_L = 2.5$, $z_s = (0,0)$, $z_L = (1,0)$

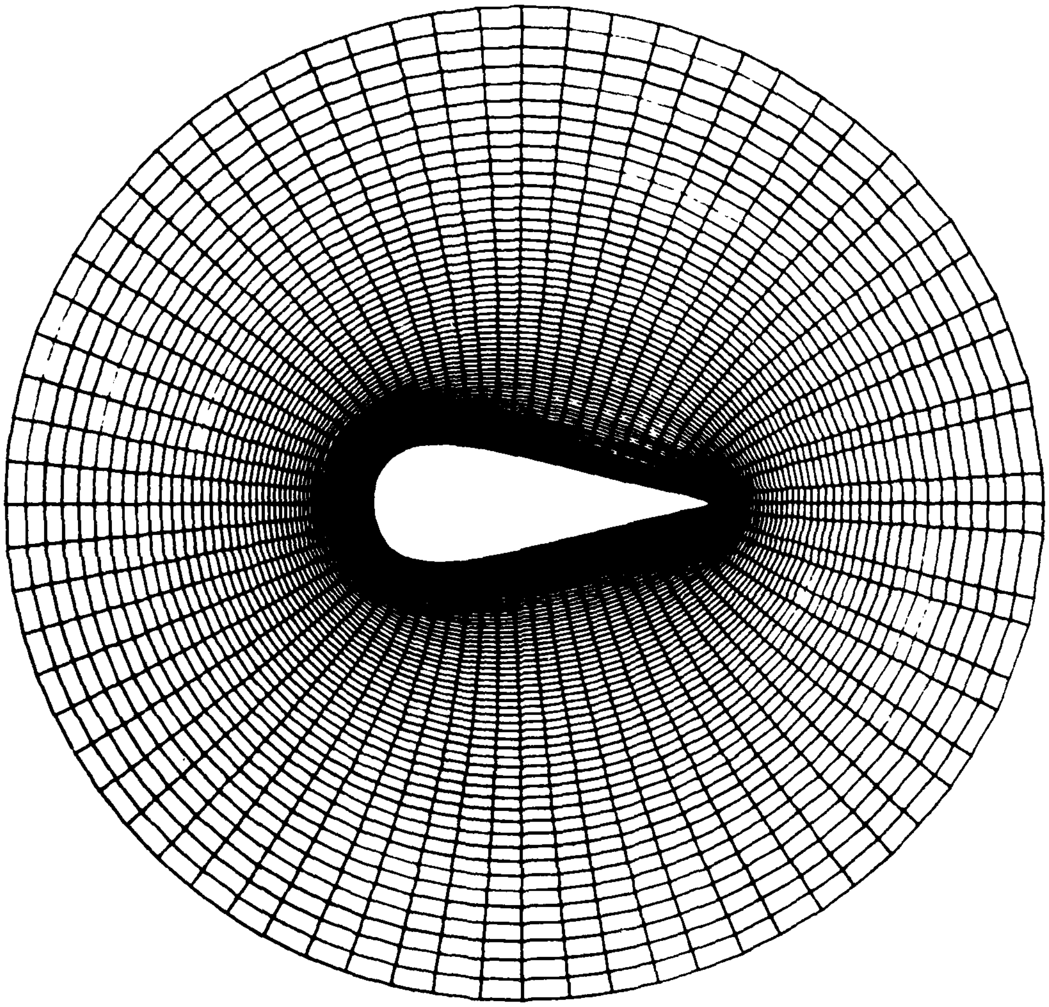

Figure 6. Joukowsky's airfoil with slightly rounded trailing edge.

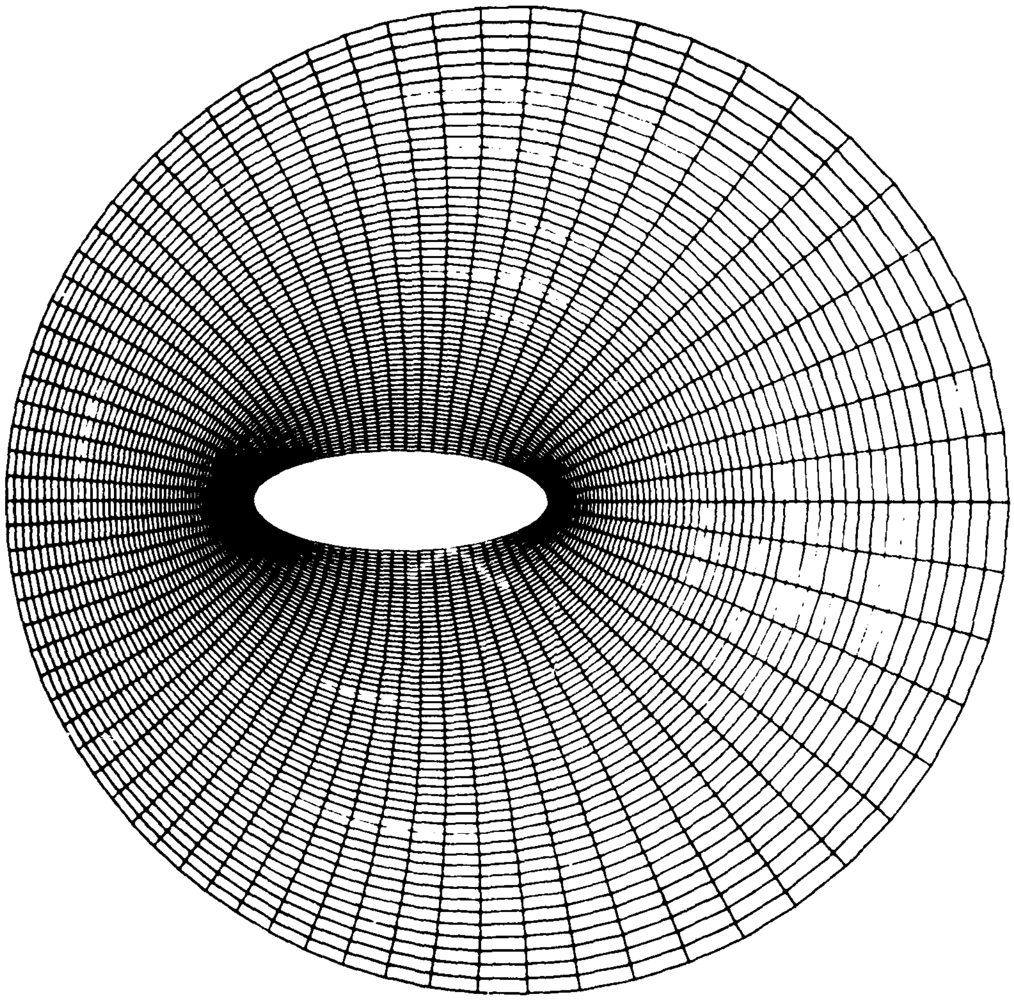

Figure 7. Non-concentric ellipses. Size data same as in Figure 3.
$z_s = (0,0)$, $z_L(1,0)$.

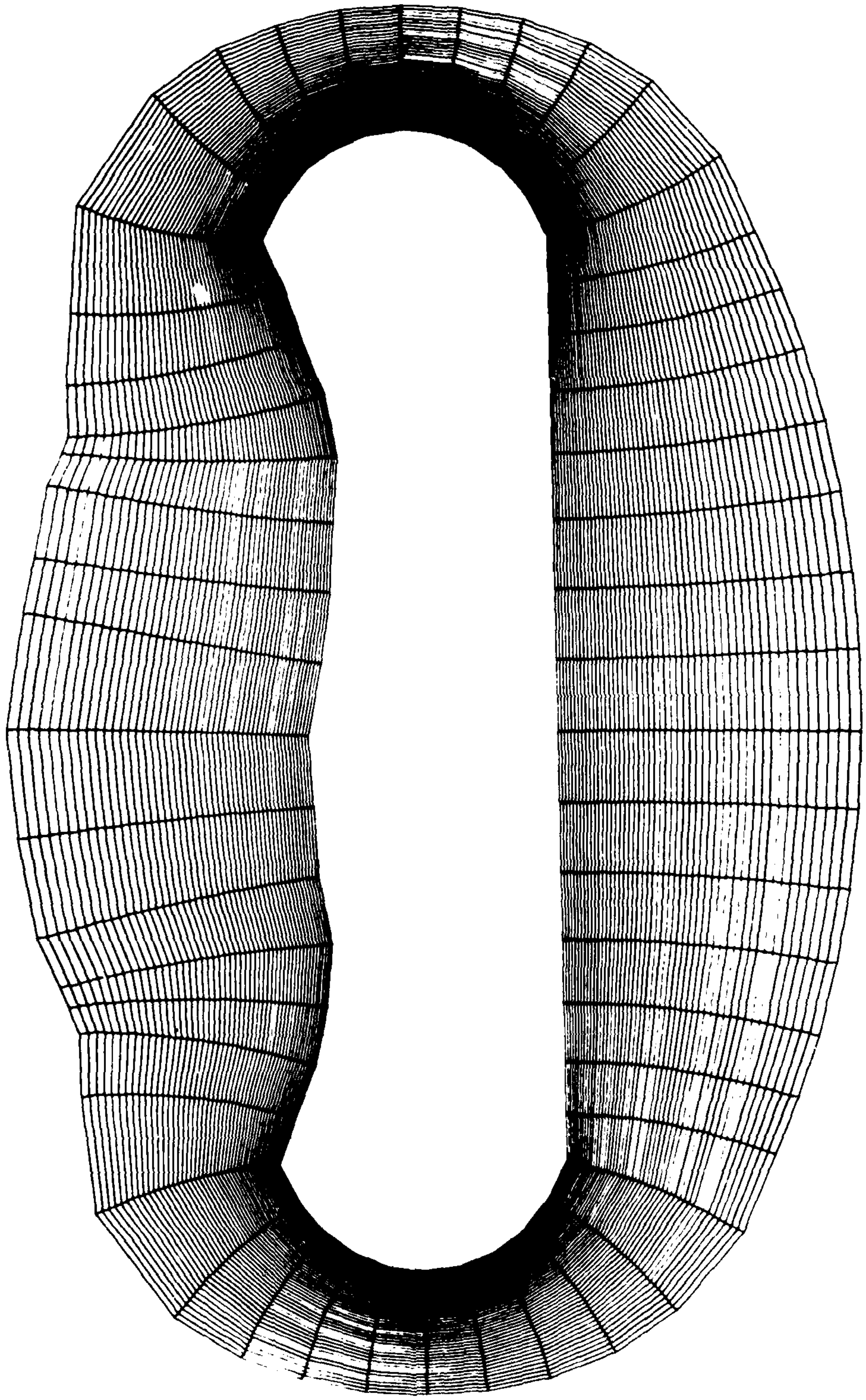

Figure 8. Generated coordinates for a body having convex, concave and straight portions. Placement of outer boundary is decided by the radius of the osculating circles of the concave portions.